Violent Relaxation of Indistinguishable Objects and Neutrino Hot Dark Matter in Clusters of Galaxies

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ABSTRACT

The statistical mechanical investigation of violent relaxation (Lynden-Bell 1967) is extended to indistinguishable objects. It is found that, coincidentally, the equilibrium distribution is the same as that obtained for classical objects. For massive neutrinos, the Tremaine & Gunn (1979) phase space bound is revisited and reinterpretated as the limit indicating the onset of degeneracy related to the coarse-grained phase space distribution. In the context of one of the currently most popular cosmological models, the Cold and Hot Dark Matter (CHDM) model (Primack et al. 1995), the onset of degeneracy may be of importance in the core region of clusters of galaxies. Degeneracy allows the neutrino HDM density to exceed the limit imposed by the Tremaine & Gunn (1979) bound while accounting for the phase space bound.

Subject headings: dark matter — galaxies: clusters: general — methods: statistical

1. Introduction

Self-gravitating astrophysical systems having initial mass and velocity distributions far from equilibrium are believed to evolve towards equilibrium by violent relaxation, a collective, collisionless process mediated by the fluctuating gravitational field. Compared to the time scale of the two-body relaxation process, the time scale of violent relaxation is short. The analytical investigation of violent relaxation in terms of statistical mechanics was first introduced by Lynden-Bell (1967). Formally, the equilibrium distribution resembles the Fermi-Dirac distribution. For stellar and galactic systems it tends towards the Maxwell-Boltzmann distribution (Lynden-Bell 1967, Shu 1978).

In this Letter we investigate violent relaxation of indistinguishable objects. We show that, despite of the difference between the violent relaxation statistics of indistinguishable and distinguishable objects, coincidentally, the equilibrium distributions of both cases are of the same form as expected by Shu (1987). When applied to massive neutrinos it turns out that the degeneracy implicit in the equilibrium distribution plays an important role leading to a reinterpretation of the Tremaine & Gunn (1979) phase space bound as a limit indicating the onset of degeneracy. In the context of one of the currently most popular cosmological models based on initial conditions compatible with inflation theory ($\Omega = 1$), the Cold and Hot Dark Matter (CHDM) model (Primack et al. 1995), the onset of degeneracy allows the neutrino HDM density to exceed the limit imposed by the Tremaine & Gunn (1979) bound while accounting for the phase space bound.

2. Violent Relaxation Statistics of Indistinguishable Objects

Following Lynden-Bell (1967) we attempt to describe violent relaxation by the use of a maximum entropy principle. The one-particle phase space μ of the objects undergoing violent relaxation is divided into a large number of microcells the dimension of which is chosen so as to yield the mass density in the units mass per spatial and velocity volume, $\Delta^3 x$ and $\Delta^3 v$, respectively. The collisionless nature of violent relaxation then imposes an additional constraint on the time evolution of the μ-space. Initially not overlapping phase elements do never overlap. Therefore, a microscopic exclusion principle for phase elements in μ-space is established. The volume of the microcells is chosen in such a way that the microcells are occupied by at most one object each and that the probability of finding occupied adjacent microcells representing two-body encounters is small (Saslaw 1985). With m the particle mass, the mass per microcell is either zero or m. Let η denote the corresponding mass density of an occupied microcell. The initial phase space distribution before violent relaxation may then be described by the set of occupied microcells.

At the macroscopic level, the microcells (or phase elements) are grouped into macrocells containing a large number ν of microcells. Let N be the total number of occupied microcells corresponding to N objects. Suppose that there is a microstate in which there are n_a occupied microcells in the ath macrocell. Due to the collisionless interaction of the violent relaxation process, the occupation state of each microcell does not change, i.e. there is no cohabitation. The indistinguishable nature of the objects under consideration leads

to a total number

$$w_a = \frac{\nu!}{n_a!(\nu - n_a)!} \tag{1}$$

of ways of assigning the n_a occupied microcells to the ν microcells of the ath macrocell as already realized by Lynden-Bell (1967) but not worked out further. Since the elements are indistinguishable, there is just one way of splitting the total of N elements into groups n_a, i.e. there are no permutations. Thus, for indistinguishable elements, the total number of microstates is

$$W_i = \prod_a \frac{\nu!}{n_a!(\nu - n_a)!} \tag{2}$$

where, for distinguishable elements, the corresponding number is (Lynden-Bell 1967)

$$W_d = \frac{N!}{\prod_a n_a!} \prod_a \frac{\nu!}{(\nu - n_a)!}\,. \tag{3}$$

The most probable state is found by the standard procedure of maximizing $\log W$ subject to the constraints of constant total energy and constant mass. Since the expressions W_i and W_d differ only by a factor $N!$, the maximization procedure performed in the continuum representation yields, coincidentally, for both expressions $\log W_i$ and $\log W_d$ the same coarse-grained phase space distribution

$$\bar{f}(v,x) = \frac{\eta}{1 + \exp[\beta(\epsilon - \mu)]}\,. \tag{4}$$

Here, $\epsilon = v^2/2 + \Phi(x)$, with $\Phi(x)$ the gravitational potential normalized as $\Phi(x \to \infty) = 0$. The Lagrange parameters β and μ are chosen according to the macroscopic constraints. Thus, the most probable coarse-grained phase space distribution of indistinguishable elements is the same as for distinguishable elements.

Shu (1978) reexamined the statistical mechanical discussion of violent relaxation in terms of particles and obtained an expression equivalent to (4) determining the occupation number n_a

$$n_a = \frac{\nu}{1 + \exp[\beta_p m(\epsilon_a - \mu_p)]} \tag{5}$$

of the ath macrocell where $\epsilon_a = v_a^2/2 + \Phi(x_a)$, and the Lagrange parameters, β_p and μ_p, now refer to the particle description. For both the cases (4) and (5), with $\bar{f} \ll \eta$ and $\nu_a \ll \nu$, respectively, the phase space distribution tends towards the Maxwell-Boltzmann distribution. Despite of these agreements, the discussions by Shu (1978) and Lynden-Bell (1967) differ what concerns degeneracy effects potentially present in (4) and (5). While Lynden-Bell (1967) considers degeneracy to be eventually important in central regions of galaxies, Shu (1978) concludes degeneracy to indicate the onset of two-body encounters. However, if two-body encounters are present, there is no microscopic exclusion principle any more. Thus the relevant phase space distribution is of Maxwell-Boltzmann type. In any case, for massive neutrinos, the discussion turns out to be irrelevant.

3. The Case of Massive Neutrinos

If the neutrino has a non-zero rest mass as predicted by some extensions of the Standard Model (Gelmini & Roulet, 1995), this may have important consequences for both the solar neutrino and the cosmological dark matter problem. Massive neutrinos with a rest mass of a few eV have been shown to account remarkably well for both problems (Primack et al. 1995). The discussion in this section is restricted to one neutrino flavor with $g_\nu = 1$.

We now apply the result of the foregoing

section to relict neutrinos of non-zero rest mass trapped in a gravitational potential, e.g. related to a cluster of galaxies. The initial phase space density of the unperturbed relict neutrino background is a relativistic Fermi distribution

$$f_\nu(p) = \frac{m_\nu^4}{(2\pi\hbar)^3} \frac{1}{e^{(pc/kT_\nu)} + 1} \qquad (6)$$

with $T_\nu = 1.95$ K (for $z = 0$ and in the units mass per spatial and velocity volume d^3x and d^3v, respectively). For $p = 0$, the maximum value $f_\nu(0) = (1/2)\, m_\nu^4/(2\pi\hbar)^3$ is obtained.

If the neutrinos are massive, the phase space distribution f of the unperturbed relict neutrino background is changed due to gravitational interactions. Violent relaxation considered here leads to a convolution of the phase space structure. Regions of initially different phase space densities end up entwined together. However, the coarse-grained phase space density $\bar{f}$ cannot exceed the maximum of the initial fine-grained phase space density f_ν since the collisionless nature of violent relaxation implies conservation of the fine-grained phase space density f.

Since neutrinos are elementary particles and their unperturbed, initial phase space distribution is the relativistic Fermi distribution (6), the volume of the microcells is chosen to be the elementary phase space volume $m_\nu^3/(2\pi\hbar)^3$. This choice guarantees the microcells to be occupied by at most one object each. The corresponding phase space mass density of an occupied microcell is $\eta = m_\nu^4/(2\pi\hbar)^3$. Note, that the initial condition assumed in the foregoing section is perfectly fulfilled by the phase space distribution of the relict neutrinos.

Suppose now the relict massive neutrinos to be subject to violent relaxation. According to the foregoing section, their coarse-grained phase space distribution $\bar{f}$ tends towards the Fermi type distribution (4). First we note, that, in general, the relation $\max\{\bar{f}\} \leq \max\{f\}$ must hold. With the natural assumption $\beta > 0$, equation (4) implies

$$(\Phi - \mu) \geq 0\,. \qquad (7)$$

where for a given β equality corresponds to the highest possible phase space density. If no degeneracy effects are present, then $\bar{f} \ll \eta$ and equation (4) requires

$$\beta\,(\Phi - \mu) \gg 0\,. \qquad (8)$$

In terms of the mass density

$$\rho_\nu = 4\pi\eta \int_0^\infty v^2 e^{-\beta\left[(v^2/2+\Phi)-\mu\right]} dv \qquad (9)$$

and the specified value of $\eta = m_\nu^4/(2\pi\hbar)^3$, inequality (8) leads to

$$\rho_\nu \ll \frac{m_\nu^4 \sigma^3}{\sqrt{8\pi^3}\hbar^3} \qquad (10)$$

where $\sigma = \beta^{-1/2}$ is the one-dimensional velocity dispersion. When solved for m_ν we get

$$m_\nu \gg \left(\frac{\rho_\nu \sqrt{8\pi^3}\hbar^3}{\sigma^3}\right)^{1/4} \doteq m_d \qquad (11)$$

This expression is interpreted as a limit on the type of the coarse-grained phase space density $\bar{f}$. For given velocity dispersion σ, neutrino density ρ_ν and a neutrino mass m_ν well above m_d, $\bar{f}$ is the Maxwell-Boltzmann distribution. On the other hand, if m_ν is of order or even below m_d, $\bar{f}$ differs from the Maxwell-Boltzmann distribution. In this case, degeneracy effects become important and the correct coarse-grained phase space

density $\bar{f}$ is the Fermi-type distribution (4). We note, that the possible onset of two-body encounters preventing Fermi-like degeneracy in the case of stars or galaxies (Shu 1978) does not apply to neutrinos. The time scale of gravitational two-body encounters for neutrinos is extremely long. Moreover, even when strong two-body encounters would occur, the Fermionic nature of the neutrinos prevents single microcells to become cohabitated. Thus, for neutrinos, a microscopic exclusion principle applies in any case.

Inequality (11) is formally equivalent to the well known Tremaine & Gunn (1979) bound on the rest mass of neutrinos which are gravitationally bound in galactic halos or galaxy clusters. However, the Tremaine & Gunn (1979) limit is derived from the assumption that $\bar{f}$ is the Maxwell-Boltzmann distribution and the condition that the maximum of the coarse-grained phase space density $\bar{f}$ cannot exceed the maximum of the initial fine-grained phase space density f given by (6). Therefore, the Tremaine & Gunn (1979) limit implicitly relies on the assumption of $\bar{f}$ to be of Maxwell-Boltzmann type. On the contrary, (11) refers to the type of the phase space distribution. Therefore we stress, that, from the point of view of (11), the limit implied should not be interpreted as limit on the neutrino mass. Instead, (11) states that the phase space distribution for m_ν near the Tremaine & Gunn (1979) bound differs significantly from the Maxwell-Boltzmann distribution.

In terms of the gravitational potential Φ and the neutrino mass m_ν it is possible to formulate several limits on the neutrino rest mass m_ν based on the phase space density $\bar{f}$ defined by (4). We intend here to derive the most robust one. According to (7) the mass density of the totally degenerate, densest state is

$$\rho_\nu = 4\pi\eta/2 \int_0^{v_l} v^2 dv \qquad (12)$$

where v_l is the limiting velocity below which the maximum number of microcells is occupied ($\bar{f} = \eta/2$) while above v_l all microcells are empty ($\bar{f} = 0$). Assuming only bound states to be relevant, i.e. $\epsilon = v^2/2 + \Phi \leq 0$, the most robust limit on the neutrino mass is found for $v_l = \sqrt{|2\Phi|}$ yielding

$$m_\nu \geq \left(\frac{12\rho_\nu \pi^2 \hbar^3}{|2\Phi|^{3/2}} \right)^{1/4} . \qquad (13)$$

This limit is robust in the sense that it holds for every coarse grained phase space distribution $\bar{f}$ representing a bound state including those eventually not described by (4), as for example anisotropic phase space distributions suggested by Madsen (1991) or Ralstone & Smith (1991). In order to compare these limits to the Tremaine & Gunn (1979) bound, the relation $\sigma^2 \approx \Phi/3$ holding for a Maxwellian velocity distribution related to a self-gravitating isothermal sphere is adopted. The Tremaine & Gunn (1979) bound on the neutrino rest mass then becomes

$$m_\nu \geq \rho_\nu^{1/4} \left(\frac{6\pi\hbar^2}{|\Phi|} \right)^{3/8} \qquad (14)$$

which is about a factor ≈ 1.18 higher than (13). In terms of the neutrino density this leads to an augmentation of the density limit of about a factor of ≈ 1.95. As a consequence, the onset of degeneracy allows the neutrino HDM density to exceed the limit imposed by the Tremaine & Gunn (1979) bound by a factor of ≈ 1.95 while accounting for the phase space bound.

In comparing the two limits, one should take into account that the Maxwellian velocity distribution related to (14) has a tail of unbounded neutrinos. For the probably more physical truncated Maxwellian distribution consisting of bounded neutrinos only, the difference between the limits would become bigger. On the other hand, the totally degenerate situation on which the limit (13) is based is unlikely to be realized by violent relaxation.

4. Discussion and Conclusion

This section considers violent relaxation of massive neutrinos in clusters of galaxies in the context of the currently popular Cold and Hot Dark Matter (CHDM) model. Extensive numerical simulations show these models to agree well with observations on cosmological scales (e.g. Klypin et al. 1993, Noltenius et al. 1994, Klypin & Rhee 1994). In what follows we refer to the two most promising neutrino mass schemes due to Primack et al. (1995). The first scheme divides the neutrino HDM in two neutrino species each with a mass of 2.4 eV while the second splits the HDM in three 1.6 eV neutrino species. The median value of the initial phase space distribution f_ν of each family of the relict neutrinos is only 0.06 (Madsen & Epstein 1984). As an approximation we therefore again take $\eta = m_\nu^4/(2\pi\hbar)^3$ where m_ν is 2.4 eV or 1.6 eV, respectively, i.e. for our purposes the phase space elements occupied by two or three neutrinos are neglected.

The left panel of figure 1 presents the contour plot of density limits derived from (13) and (10) for the 1.6 eV neutrino mass scheme. The right panel shows the density limits for the 2.4 eV mass scheme. The limits are plotted as upper limits to the total matter densities ρ consisting of HDM, CDM and baryonic matter. They are functions of the gravitational potential depth, indicated by the velocity dispersion of the corresponding self-gravitating isothermal sphere $\sigma \approx \sqrt{\Phi/3}$, and the ratio of the neutrino HDM density ρ_ν to the total matter density ρ. The solid contours represent the most robust limit for bound states given by (13). Along these contours of specified total density ρ, the neutrino HDM is found in the totally degenerate, densest state. Augmentation of ρ is only possible if it is accompanied by a change of the ratio ρ_ν/ρ or σ. On the other hand, the more ρ_ν/ρ is diminished while leaving ρ and σ unchanged, the more the coarse-grained phase space $\bar{f}$ of the neutrino HDM becomes non-degenerate. The dashed contours are the density limit obtained from (10). Well below this limit, $\bar{f}$ becomes a Maxwellian. In addition, this density limit by (14) corresponds to the Tremaine & Gunn (1979) bound.

The parameter range of figure 1 includes typical values of σ and ρ_ν/ρ expected for clusters of galaxies. The suppression of the HDM neutrino density ρ_ν compared to the CDM and baryonic matter density due to its lower density contrasts (e.g. Klypin et al. 1993, Noltenius et al. 1994) is reflected by values of ρ_ν/ρ below the cosmological value $\Omega_\nu = 0.2$ (Primack et al. 1995). For the 1.6 eV neutrino mass scheme (left panel of figure 1), one observes a broad parameter region yielding upper total density limits comparable to typical core densities of cluster of galaxies. Adopting the 2.4 eV neutrino mass scheme (right panel of figure 1) shrinks the relevant parameter region. However, since the coarse-grained phase space distribution $\bar{f}$ of the neutrino HDM is expected to be a Maxwellian only well below the limit indicated by the

dashed contours, it is found that for both CHDM models $\bar{f}$ may differ significantly form the Maxwell-type distribution. Accordingly, $\bar{f}$ should be represented by the Fermi-type distribution (4). Due to (7) it accounts for the phase space bound while it allows the neutrino density to exceed the density limit imposed by the Tremaine & Gunn (1979) bound (indicated by the dashed contour lines, too) by a factor of ≈ 1.95.

Violent relaxation eventually fades before the final state (4) is attained. Thus, (4) is unlikely to be reached throughout the whole cluster, but it is reasonable to hold in the central region where violent relaxation occurs most violently. In order to determine the global properties of self-gravitating spheres based on coarse grained phase space distributions (4), a careful investigation is needed. In particular, the onset of degeneracy eventually related to core formation and not yet observed in simulations (Bryan et al. 1994) appears interesting.

In summary, the statistical mechanical investigation of violent relaxation has been extended to indistinguishable objects. It is found that the equilibrium distribution is the same as that obtained for distinguishable objects. However, in contrast to stellar and galactic systems, the onset of degeneracy is not prevented by two-body encounters in the case of massive neutrinos. Thus, the coarse-grained phase space density $\bar{f}$ is of Fermi-type and, to some extent, degeneracy may be present. For currently popular CHDM models, the onset of degeneracy is shown to be relevant in the core region of clusters of galaxies.

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Fig. 1.— Contour plot of upper limits to the total density ρ (in kg m^{-3}) for the 1.6 eV (left panel) and the 2.4 eV (right panel) neutrino mass scheme. The limits are plotted as functions of the gravitational potential represented by the one dimensional velocity dispersion σ according to $\sigma^2 \approx \Phi/3$ and the ratio ρ_ν/ρ. The solid contour lines represent the most robust density limit for bound states. Dashed contour lines indicate the density limit related to the type of the coarse-grained phase space distribution $\bar{f}$ of the neutrino HDM. Well below this limit $\bar{f}$ is a Maxwellian.

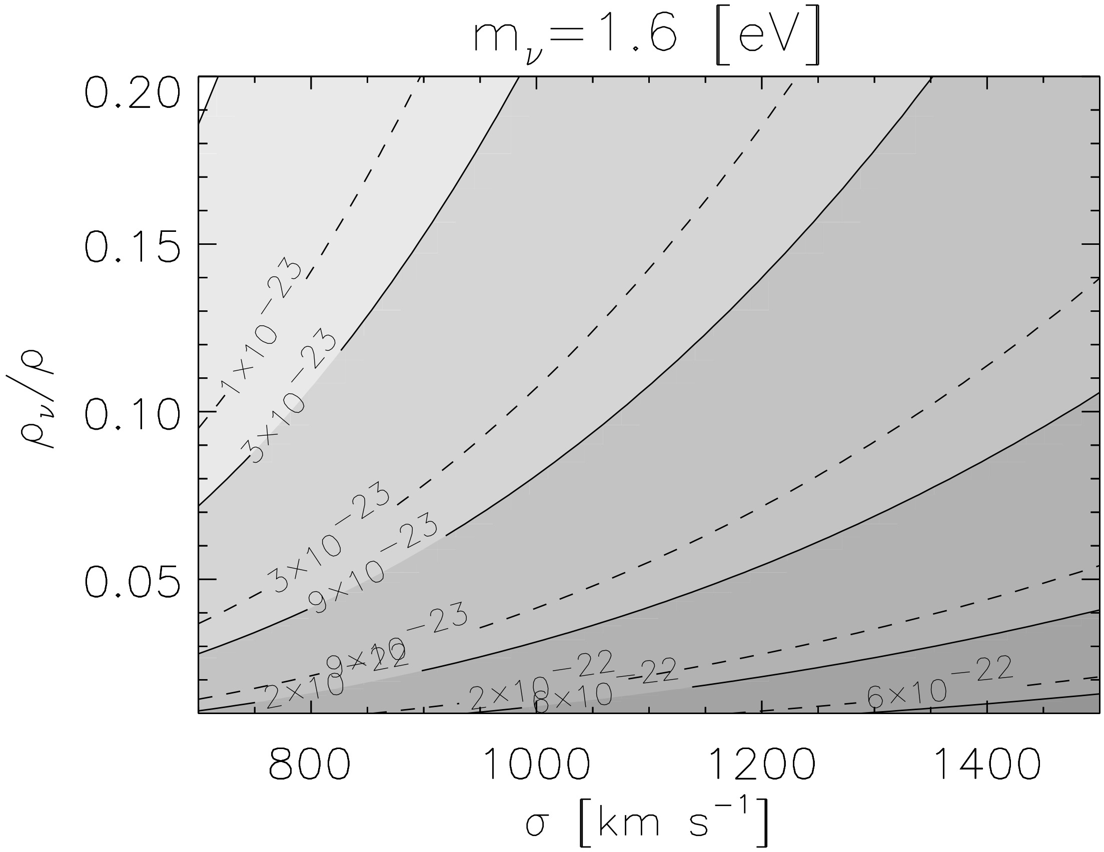

Fig. 1.- (left panel)

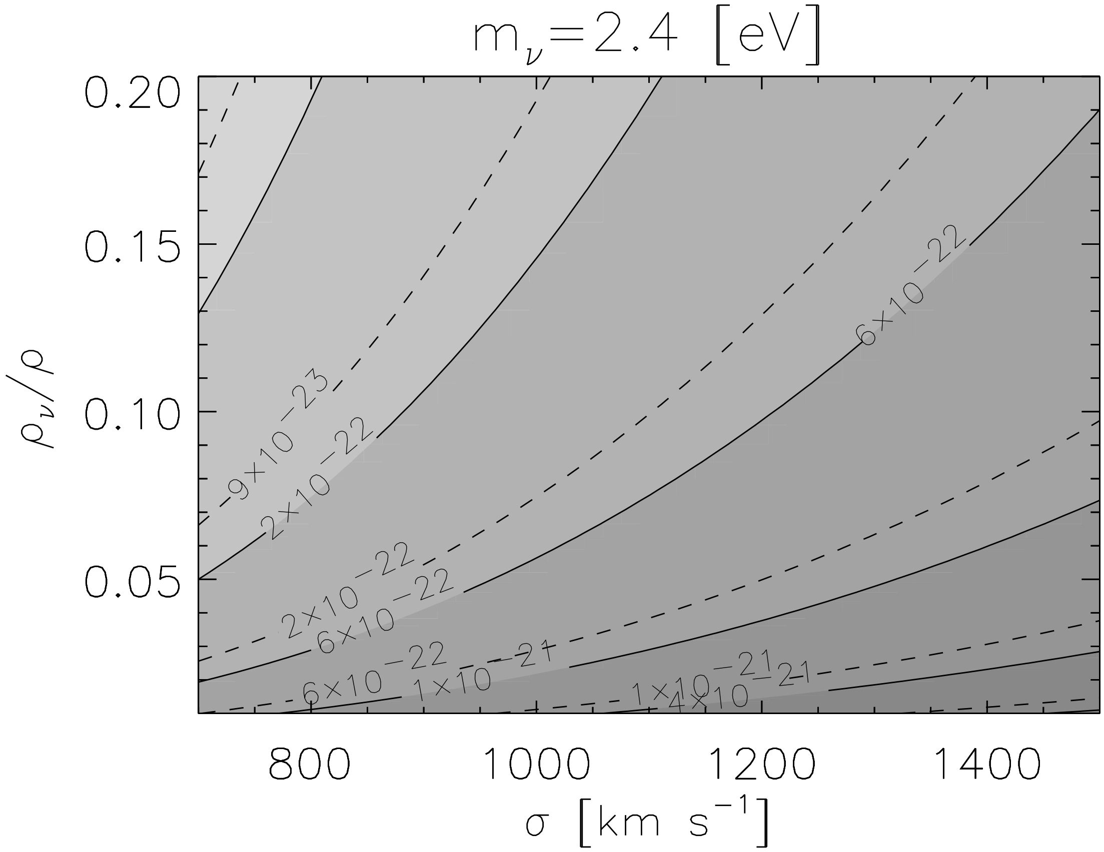

Fig. 1.- (right panel)